

Running head: MAKING SENSE OF THE TOTAL OF TWO DICE

The Emergence of New Situated Internal Resources

for Making Sense of the Total of Two Dice

Dave Pratt

University of Warwick, U.K.

Author Note

This article is based on aspects of my doctoral research (Pratt, 1998), available from my home page:  
[http://fcis1.wie.warwick.ac.uk/~dave\\_pratt/](http://fcis1.wie.warwick.ac.uk/~dave_pratt/)

I would like to acknowledge Professor Richard Noss for his help and support throughout this study.

Correspondence concerning this article should be addressed to: Dr. D. C. Pratt, Mathematics Education Research Centre, Institute of Education, University of Warwick, Coventry, U.K. CV4 7AL or by eMail to: [dave.pratt@warwick.ac.uk](mailto:dave.pratt@warwick.ac.uk)

## Abstract

Many studies have shown that the strategies used in making judgements of chance are subject to systematic bias. This study examines an associated, but sparsely researched, area of paramount importance to education. Little is known about the relationship between the external structuring resources, made available for example in a pedagogic environment, and the construction of new internal resources, such as intuitions, about chance and randomness. This study uses a novel approach in which young children articulated their meanings for chance through their attempts to 'mend' possibly broken computer-based stochastic gadgets. Here I describe the interplay between informal intuitions and computer-based resources as the children construct new internal resources for making sense of the total of two spinners and two dice.

## The Emergence of New Situated Internal Resources

### for Making Sense of the Total of Two Dice

#### Introduction

. In this study, when children used a specially designed computer-based microworld, their intuitions were exposed and I was able to observe emergent sense-making for compound stochastic events. The results shed light on a sparsely researched area concerning the nature of how the resources in a pedagogic setting might shape children's intuitional knowledge.

There are many everyday situations where the external structuring resources might support the construction of primary intuitions (Fischbein, 1975) for randomness --for example lotteries, board games, betting and sports. In fact the language used to mediate such activity extends into many other aspects of our lives. People meet in 'chance encounters'; success or failure is often put down to 'just chance'; a decision is '50/50'; the weather forecaster claims a '75%' chance of rain. One might suppose then that, given a multitude of opportunities and a natural language on which to build an intuitional basis for randomness, probability would be a relatively easy topic to learn. The opposite appears to be true. There is a sort of folk-lore amongst teachers which suggests that probability is extremely difficult to learn. It has been proposed that practical approaches to the teaching of probability would enable the development of secondary intuitions (Fischbein, 1982):

For instance, in order to create new correct probabilistic intuitions the learner must be actively involved in a process of performing chance experiments, of guessing outcomes and evaluating chances, of confronting individual and mass results *a priori* calculated predictions, etc. New correct and powerful probabilistic intuitions cannot be produced by merely practising probabilistic formulae. (p12)

To some extent, in the modern mathematics movement, teachers adopted such methods and probability became an important part of the curriculum. Lack of success has led to an avoidance of probability in the curriculum or a return to formal approaches which intentionally sub-ordinates their students' intuitions. One explanation for this story of failure could be that probability is intrinsically difficult; another explanation might be that our knowledge of how to support normative stochastic intuitions is as yet too impoverished to develop effective learning environments. I wish to examine these two explanations with a particular focus on the total of two dice.

### The Total of Two Dice? 'It's Just Chance.'

One school of thought would argue that a sophisticated knowledge of the total of two dice depends upon mental schema which have typically not yet developed at, say, age 10 or 11. In Piaget and Inhelder's vision (1951), the construction of probabilistic knowledge about stochastic events emerges out of the need to transform irreversible random mixtures into deductive operational knowledge as apparent when they write:

As soon as chance is discovered as indeterminate relationships not composable by operative methods and, contrary to all the operations, irreversible, the mind seeks to assimilate this unexpected obstacle encountered on the road to developing deduction. ... It is from this need that probabilistic composition is born. (p. 230)

Knowledge about compound events in particular is contingent upon the construction of both combinatorial and proportional mental systems. It seems, for example, non-contentious that a proper grasp of the probability of obtaining a 7 with two dice involves knowing how many configurations of two dice result in a total of 7, to which the probability of obtaining a 7 is directly proportional. According to Piaget and Inhelder, the schema for such understanding are late developments requiring the emergence of formal operations. In their analysis, the organism responds to the need to treat the stochastic as if it were reversible through a developmental process they call accommodation. A central theme of this paper is to question whether the accommodation of compound events is in fact essentially developmental, independent of the tools and resources available for the expression of the phenomena the child is trying to describe.

In the absence of well-developed schema for probabilistic knowledge, we might expect to observe a range of intuitive strategies (often referred to as heuristics) for dealing informally with situations which might otherwise be formally analysed in terms of probability. Over recent decades, a substantial research effort has identified a range of strategies. For example, in the availability heuristic (Tversky & Kahneman, 1973), judgements are made by the evocation from memory of similar events. Bias is introduced, they argue, because such memories are largely dependent on salience rather than on frequency. The availability heuristic is apparent when children regard certain throws of two dice as special. The frustration of failing to gain the required double six to begin a board game is likely to become imprinted in memory, rendering that throw 'less likely' than say a double three. Another example of a strategy used to make judgements of chance is the representativeness heuristic (Kahneman & Tversky, 1973). A particular case of this heuristic (referred to by Kahneman and Tversky as local representativeness) is observable when people believe that a sequence of events generated

stochastically will represent the essential characteristics of that process, even when the sequence is quite short. Accordingly, for the total of two dice, all 11 possible outcomes should appear even in relatively short sequences of results.

Kahneman and Tversky's work was based on experiments with adults, often university students. This evidence that even adults employ heuristics which contain systematic bias supports the notion that probabilistic concepts are intrinsically difficult. Another bias, particularly relevant to this study, has been termed the equiprobability bias, a tendency to assume that different outcomes are equally likely. In the case of throwing two dice, this bias might be articulated as 'different totals are equally easy to obtain -- it is just a matter of chance'. LeCoutre (1992) carried out an experiment using three square cards showing triangles on two of the cards and a square on the third. The shapes were configured in such a way that subjects could generate a house shape (in two ways) or a rhombus (in one way) by placing pairs of cards together. The subjects (15 to 17 year olds) were shown how the house and rhombus shapes could be constructed and were asked whether a house was more, equally or less likely than a rhombus. The same subjects were then asked a similarly structured question where two orange flavoured and one lemon flavoured candy were placed in a bag and the subjects were asked to compare the likelihood of choosing various combinations of pairs of candy. Finally the subjects were asked to compare the two experiments. The results showed the number of correct responses diminished markedly from question 1 to question 2. LeCoutre argued that the equiprobability bias was resistant to modification (even amongst individuals grounded in probability theory) but that a correct response could be induced by masking the chance element of the problem. She concluded that correct cognitive models are often available but are not spontaneously associated with the situations at hand.

Konold (1989) has suggested that some college students use a non-probabilistic method, the outcome approach, in which the task is seen as one of predicting the particular outcome rather than judging its likelihood. A focus on outcome might lead to all totals of two dice being seen as equal in the sense that it is not possible to predict which outcome will appear. Konold et al (1993) asked high school students and undergraduates to select the most likely outcome of tossing a fair coin five times. They were offered five options; the first four proposed particular sequences of heads and tails whilst the fifth suggested that all of the previous four were equally likely. They found evidence supporting the outcome approach. A second task asked the same subjects to choose which of the five options was least likely. Now there was evidence that some subjects gave responses, thought to be based on the representativeness heuristic, inconsistent with their first responses. By altering the wording of

the question, the researchers induced inconsistent responses.

Watson et al (1997) reinterpreted the data in the study by Konold et al to find them consistent with a developmental model for learning concepts in probability, based on the multimodal functioning SOLO model (Biggs & Collis, 1982). This model was constructed by asking three multiple choice questions to children in each of Grades 3 (322 children), 6 (310) and 9 (382). The first question asked the subjects to consider rolling one six-sided die. They had to respond with an explanation as to whether a one or a six were easier to throw or whether both were equally easy to throw. The second question considered a hat containing 13 boys' and 16 girls' names. The subjects had to decide whether it was more likely that a boy's or a girl's name would be chosen or whether they were equally likely. In the third question, Box A was said to contain 6 red and 4 blue balls and Box B contained 60 red and 40 blue balls. The subjects were asked whether Box A or Box B should be chosen if they wanted to obtain a blue ball or whether it didn't matter. The model developed from the data identifies four levels of responses: ikonik, where decisions are based on non-mathematical beliefs (such as luck or favouritism), unistructural, where uncertainty is recognised, multistructural, where some elementary quantification occurs and relational, a correct quantification occurs or a relationship is established. The inconsistency in the students' responses in the study by Konold et al was seen as typical of unistructural functioning, when, in more complex settings, responses offer single ideas without justification and in which potential conflict is not recognised. Watson et al have stated an intent to apply their model to the total of two dice but these experiments have not yet been reported. One might conjecture that unistructural responses will similarly be characterised by such inconsistencies and lack of justification, whereas the higher level of multistructural responses might reflect a need to resolve such conflicts and an emerging use of combinatorial thinking.

Another study (Fischbein and Schnarch, 1998) is also seeking to isolate a developmental model for probabilistic intuitions. They began with a hypothesis, based on the study of intuitions of infinity, that probabilistic intuitions would stabilise with the onset of formal operations at about 12 years of age. They investigated the intuitions of twenty students in each of Grades 5, 7, 9 and 11 and eighteen college students. They were asked seven probability questions directed specifically at well-known misconceptions. They found that a misconception associated with the representativeness heuristic decreased in frequency with age whereas a misconception thought to be due to the availability heuristic increased with age. Even more strangely, the equiprobability bias was remarkably stable across all ages. In this latter case, the subjects were asked to

consider rolling two dice simultaneously. They were asked whether the pair 5-6 or the pair 6-6 were more likely or whether they were both the same chance (other responses were also allowed). In Grade 5 70% gave responses consistent with the equiprobability bias. For Grades 7, 9, and 11 the percentages were 70, 75 and 75 respectively. Even amongst the college students, who were training to be teachers and were specialising in mathematics, the percentage was 78. The author's chose at this stage not to analyse the subjects' explanations, whose interpretation was seen as complex and left for a later stage of the ongoing research. We do not know therefore how prevalent the equiprobability bias actually was in the subject's answers or whether for example some subjects were confused how to interpret 'the pair 5-6'. In their brief report, the authors conjectured that the misconceptions are associated with general schemata. Accordingly, in simple problems, the general schemata may be adequate to address the problem and so the frequencies of the respective misconception diminish as the schemata become better integrated. In more complex situations, they argue, the general schemata may be inadequate to deal with the specific constraints of the problem and the frequencies of the misconception increase with the student's age.

The above studies have adopted a methodology in which snapshots are taken of how subjects respond to specific problems and tasks. The aim of this methodology has been to observe a momentary decision and to deduce the mechanisms used to come to that decision. When the snapshots are taken at many different ages, the aim has been to identify developmental trends in those mechanisms. Everyday settings may well involve instantaneous decision-making of this type where the immediate environment is relatively impoverished in terms of ready-to-hand tools, which might influence that decision-making. In contrast, we have little knowledge about the process by which such biases might be constructed and how new ways of thinking might be generated. This gap in the research is extremely important to the classroom context. The conclusions from this research on biases do not illuminate situations where tools are provided with the explicit intention of shaping thinking in change as is the case in pedagogic settings. LeCoutre's data suggests that even when we have normative ways of thinking about stochastic situations, they are not necessarily cued. We need to understand the relationship between the context and which ways of thinking are cued. Experiments, suggesting that the equiprobability bias is resistant to change, have studied the momentary impact of changes in the experimental conditions, for example in the problem context. One study (Tarr, 1998) has presented evidence that a particular system of instruction reduced the frequency of the equiprobability bias quite dramatically. In this study 26 Grade 5 students, undergoing a teaching programme in two groups, were compared to 13 students in a control group. Initially and during the instruction, the students were observed to be misusing the phrase '50-50' in two

different ways. They applied the phrase to situations in which all (more than two) events were equally likely and also to situations in which there were events of unequal probability. After the instruction, Tarr found that the use of the equiprobability bias was far less frequent. This report suggests that there may be learning environments which cue more normative ways of thinking. Tarr's report provides data before and after the instruction but he has not yet reported the data generated during the instruction. If we are able to articulate the connection between the structuring resources and the thinking in change, we might better develop pedagogic approaches which not only tend to support the construction of intuitions less subject to bias but also signal how connections between such intuitions and formal probabilistic knowledge might be encouraged.

In this study, I explicitly focus on the relationship between the external resources, designed with the intent of provoking changes in stochastic thinking, with the construction of internal resources for randomness by observing the forging of connections between the external and the internal with respect to compound stochastic events. In so doing, I aim to illuminate the notion of webbing which Noss and Hoyles (1996) characterise as follows:

- it is under the learner's control;
- it is available to signal possible user paths rather than point towards a unique, directed goal;
- the structure of local support available at any time is a product of the learners' current understandings as well as the understandings built by others into it;
- the global support structure understood by the user at any time emerges from connections which are forged in use by the user. (p108)

The notion of webbing places emphasis on the interaction between internal and external resources. I use the term internal resources in the same way as do Noss and Hoyles; the term incorporates all forms of knowledge that might be held by an individual, encompassing formal and intuitive knowledge, available for use in sense-making activity. One particular form of internal resource, a situated abstraction, proves to be of particular importance in this study. This is a type of intuition which emerges through sense-making activity and is articulated in situated specific terms. The resource is an abstraction in so far as the child has through the sense-making activity internalised a heuristic which can be applied to various scenarios within that context. The situated abstraction may have potential to be adapted to other contexts but this is not necessarily the case and even if it is, the child may be unaware of this potential. Noss and Hoyles refer to situated abstractions as

follows:

We intend to use the term *situated abstraction* to describe how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed (p. 122)

The notion of webbing immediately suggests a multiplicity of internal resources and I now wish to turn my attention to this aspect of the framework which will underpin this study.

### Multiply Concurrent Resources

Let me review where I have reached in my argument. There is a significant gap in our knowledge about children's internal resources for making sense about randomness. Although a catalogue of biases have been identified in the ways of thinking adopted by people when making judgements of chance, we know little about the relationship between the specificities of external resources, designed to provoke changes in stochastic thinking, and the emerging internal resources for sense-making. In fact there is every reason to believe that children and adults hold a multiplicity of internal resources simultaneously.

In presenting this argument, I am heavily influenced by the situated cognition movement, and in particular by the work of Lave (1988) and Nunes, Schliemann, and Carraher (1993), who have argued that the structuring resources in a setting and knowledge are dialectically related; the setting provides meaning for knowledge in such a deep way that the knowledge is somehow embedded in the setting<sup>1</sup>.

If we accept this position, then we are drawn inevitably to the conclusion that the biases discussed above are situated in the sorts of experiences that people encounter in the everyday. In this view, board-games become an environment in which these biases are originated, or at least reinforced and given meaning. Different experiences might support the generation of new resources. By designing tools which provide resources not available in conventional everyday experience, it might be possible to observe the construction of new internal resources for the total of two dice.

Is it reasonable to suppose that we hold more than one resources for making sense of the total of two dice? In fact, it is not unusual for studies of probabilistic thinking to find inconsistencies in what children say about stochastic events. Such inconsistencies can sometimes be so juxtaposed that it is hard to draw any conclusion other than that, whatever internal resources led to those contradictory articulations, they must have been

different and they must have been in existence simultaneously. In two studies, eighth grade children (Vidakovic, 1998) and pre-service teachers (Speiser & Walter, 1998) have both been observed to articulate two separable explanations for the behaviour of two dice. In one suggestion, pairs of dice scores were treated as unordered (for example, according to this meaning, 1 and 3 was indistinguishable from 3 and 1) resulting in a 21-element possibility space. An alternative suggestion, consisting of a 36-element possibility space, was also expressed. Both the papers discuss how the thinking behind these contrary suggestions was negotiated. Konold, Pollatsek, Well, Lohmeier, and Lipson (1993) argue that apparent conflict in the eyes of the expert may simply not exist for the child who can simultaneously hold supposedly inconsistent views:

one way to produce conceptual change is to create situations for which answers based on a particular incorrect intuition produce cognitive conflict. ... The results of the present study suggest one limitation to the cognitive-conflict approach — a situation designed to contrast normative with informal reasoning may produce no conflict. (p. 412)

I find echoes between these observations and the work on conceptual change by diSessa (1993). By observing Physics students, grappling with Newton's Laws of Motion, diSessa has constructed a model for conceptual change based around the notion of phenomenological primitives, or p-prims for short. These p-prims are a collection of heterarchical and rich ways of seeing and sometimes explaining the world. Some are compatible with formal physics and so encouraged, thus taking on higher priority, others are not so. P-prims are not in themselves laws of physics but serve a variety of purposes such as heuristic cues to more specific technical analyses. Some p-prims lose status, being cut apart, explained in terms of higher priority ideas.

P-prims are relatively minimal abstractions of simple common phenomena. Physics-naive students have a large collection of these in terms of which they see the world and to which they appeal as self-contained explanations for what they see. In the process of learning physics, some of these p-prims cease being primitive (and are seen as being explained by other notions), and some may even cease being recognised at all. But many become involved in expert thought in very particular ways. (diSessa, 1983, p. 32)

In diSessa's model of conceptual change, we might imagine the biases described above as direct abstractions from experience. They may be akin to diSessa's p-prims or at least closely connected to such primitive meanings. diSessa's model suggests though that it should be possible to provide new experiences from which new resources might be abstracted. These new ways of making sense of phenomena would not replace existing resources but may be modifications of current resources or they may live alongside the older resources; by

offering further experiences, the new resources may gradually take on higher priority by providing a more consistent view of phenomena than those prior resources with their in-built bias.

With diSessa's model of conceptual change in mind, I can express the aims of this study as three questions articulated quite explicitly in the language of webbing.

- (i) What are the internal resources which children use to make sense of the total of two dice?
- (ii) In making sense of the total of two dice, what situated abstractions are forged through the webbing of these internal resources and the external resources embedded in the setting?
- (iii) What are the features of the webbing process which determine the extent to which these situated abstractions become tools for the forging of new connections in related activity?

It will become apparent in the forthcoming data how the children in this study seemed to hold concurrently a variety of internal resources for making sense of the total of two dice and that these resources might sometimes be seen as conflicting, but which became 'tuned towards expertise' (in diSessa's terms) through interaction with the computer-based tools and resources. I claim that this tuning took place through the webbing between the original internal resources and the external computer-based resources, and involved the construction of situated abstractions, dependent for their meaning on the specificities of the setting.

In the next two sections, I describe the method for the current study and the main features of the computer-based tools provided in the form of the Chance-Maker microworld, concentrating on those aspects relevant to the study of children's evolving meanings for the total of two spinners and two dice.

### Approach

My aim was to construct a setting in which individuals would express their beliefs as formal conjectures in a symbolic (programming) medium, and so be able to examine the consequences of those beliefs. I hoped that learners would, as a result of these experiences, reconstruct their beliefs. I will outline the relevant aspects of the microworld design later. The computer setting facilitated the detailed tracing of children's intuitions because the activity within the computational medium required the articulation of those intuitions. In this sense, the computer acted as a window (Noss & Hoyles, 1996) on the children's internal resources for stochastic sense-making.

The methodology was founded on an iterative design in which the study of the children was carried out alternately with the modification and development of computer-based tools and resources. In each iteration, I worked with a new group of children. It turned out that the study of the children's resources became increasingly systematic and focused; the tools converged on a design exposed both the children's internal resources, brought to the activity at its outset, and the changes as new connections were forged during activity. Convergence in the iterative design process manifested itself as a decreasing need in practice to make substantive changes to the microworld in response to the tool-usage phase of the previous iteration. Such convergence provided some support for the reliability of prior design decisions.

The final iteration involved 16 children, aged ten and eleven years of age. Each child was interviewed individually using a semi-structured schedule in which the children articulated their ideas about randomness before working with the computer-based tools. The early interactions with the tools provided further insights into the nature of those initial resources. For the sessions with the computer, the children worked in pairs for between 2 and 2.5 hours. The sessions were conducted as clinical interviews with myself acting as participant observer, interacting with the children in order to probe the reasons behind their actions, later interpreting these reasons in the light of observations based on other children's work. In general, the aim was to allow the children to be in control of their explorations, making decisions and moving in directions of their own choice. Most interventions were intended to be neutral with respect to changing the direction of the children's thinking. Typically such interventions were concerned with explaining technical matters or probing what lie behind a child's actions. In contrast, occasionally interventions were intentionally experimental and so non-neutral. They sought to make some change in the direction of the activity with possible implications for conceptual change. Such interventions were occasionally needed because the children were clearly stuck or embarked upon a path with no potential pay-off from either the research or learning perspectives. Experimental interventions were also used to explore whether children were able to work with a new idea, addressing the question, 'What is the maximal level of performance that the child's internal resources, supported by the computer's tools, can achieve?' Such interventions were not often relevant and were only used when a child seemed to be particularly confident and already performing with some fluency.

The actions of the children within the computer environment were video-taped directly and the discussions were audiotaped and superimposed on the videotape. From the transcripts, I developed case accounts, plainly told stories, avoiding as far as possible interpretations of the transcript. From these plain case accounts, I

developed interpretative case analyses, in which various inferences were made as to why and how the children's internal resources were modified. The case analyses drew on the transcripts of the pre-interviews as well as the case accounts of the clinical interviews. These case analyses were made available for a colleague and differences of opinion about the inferences were discussed and resolved.

A trace of each pair's work was developed. This trace indicated the path taken including the situated abstractions articulated during that journey. As an illustrative example, I include below (Table 1) the trace of two children's (Anne and Rebecca) interactions with the TWO-SPINNERS gadget. It may be useful for the reader to return to this trace when reading later the detailed account of Anne and Rebecca's work. (The references refer to the paragraphs in the case analysis.)

Ref.	Internal Resources (SA = situated abstraction)	Critical Interventions
6.10.2 to 6.10.3	Two spinners in everyday contexts are fair and so the totals are equally likely. The number of ways the totals are represented in the workings controls the size of the sectors in the pie chart (provided the number of trials is large) (SA)	
6.10.4 to 6.10.7	<i>Activity: A &amp; R do 1000 new trials. The pictogram shows that 5 is missing. They edit the workings to include 2+3 and 3+2. They repeat a new 1000 trials. The pie chart is uneven. A &amp; R debate whether the doubles should be written twice into the workings. Anne's argument, that putting 1+1 in again, but not 2+2, would equalise the pie chart, persuades Rebecca. They edit the workings to include another 1+1. They repeat 1000 new trials. The pie chart looks even.</i>	
6.10.7		You have put 1+1 in twice but 2+2 only once. Is it fair to put 1+1 into the workings twice?
6.10.7 to 6.10.8	<i>Activity : A &amp; R edit the workings to exclude one of the 1+1's and they include 1+3. They repeat 1000 new trials and the pie chart shows most 4's and least 2's and 6's.</i>	
6.10.9 to 6.10.10	4 is easier to get than the other totals because there are more ways of making 4 (SA) 6 and 2 are harder to get because they have less ways in the workings, but 6 is even harder because the sector in the pie chart is smaller than that for 2 (SA)	Repeat the experiment.
6.10.10	<i>Activity: A &amp; R repeat 1000 new trials. The pie chart is similar except that the 6's are more frequent than the 2's.</i>	

6.10.11	6 is the same as 2 (SA) 4 is easier to get than 2 for non-computational spinners as well as for computer gadgets (SA)	
---------	--	--

Table 1 : A trace of Anne's and Rebecca's construction of resources for TWO-SPINNERS.

The case analyses and traces were studied and compared across the eight case analyses to identify consistencies and marked differences. Consistency of an issue from case to case provided a check on reliability whilst variations suggested the limitations. These issues were the subject of seminar discussions within a critical audience of research workers. I will describe one such issue that had emerged during earlier iterations and became very important in the final iteration. This was the separation of internal resources for stochastic sense-making into two categories, local and global.

Stochastic phenomena exhibit certain behaviours in the short term. The resources that we can construct from this short term behaviour concentrate on the here and now, local in the sense that such resources focus on trial by trial variation. Local resources have a number of characteristics:

- The next outcome is not predictable, though some apparent success in the short term may be experienced (unpredictability).
- No patterned sequence in prior results is sustainable over the longer term (irregularity).

These two resources are closely related. Though the former looks to the future whilst the latter considers historical data, predictions are often based on previous experience. A third characteristic of local resources is tied into the observer's own influence over events:

- The observer is unable to exert physical control over the outcome of the phenomenon (unsteerability).

By physical control, I refer to factors which give you a direct feeling of control, like how you throw the dice, or toss the coin. On the computer, this sort of control can only be realised through a device such as a mouse or a joystick. The main point here is that, if I believe that I can direct the outcome through my own physical actions, then I am unlikely to regard the phenomenon as stochastic.

Global resources focus on an aggregated overall view of the stochastic and are characterised by the following properties:

- The proportion of outcomes for each possibility is predictable (probability).
- The proportion of prior results for each possibility in the possibility space will stabilise as an increasing number of results is considered (large numbers).
- The observer is able to exert control over these proportions through manipulation of the possibility space (distribution).

There is an interesting correspondence between each set of resources. The most striking aspect of this correspondence is that each local resources is inverted in relation to its global counterpart. Thus, unpredictability as a local resources is inverted in comparison to the global resource of predictability (in a proportional sense). Similarly, control can not be exerted locally, whereas there is a global resources for control through manipulation of the distribution. In order that the subsequent data and its analysis is meaningful to the reader, I will now describe a small portion of the domain of stochastic abstraction which emerged through the iterative design process and became known as the Chance-Maker microworld.

#### The Chance-Maker Microworld

The software tools were provided within a Boxer<sup>ii</sup> environment. Boxer has evolved out of Logo and provides a powerful environment in which children are able to express their mathematical ideas in computational terms. The central objects of the microworld are a series of gadgets, computational tools which behave in many identifiable respects like their everyday counterparts. The gadgets were designed as culturally familiar devices so as to maximise the likelihood that their behaviour and appearance would cue intuitions and expectations based on their everyday equivalents. Figure 1 depicts the range of gadgets<sup>iii</sup> -- coins that can be tossed and coins that roll, spinners, dice, and a Frisbee -- that had emerged by the final iteration.

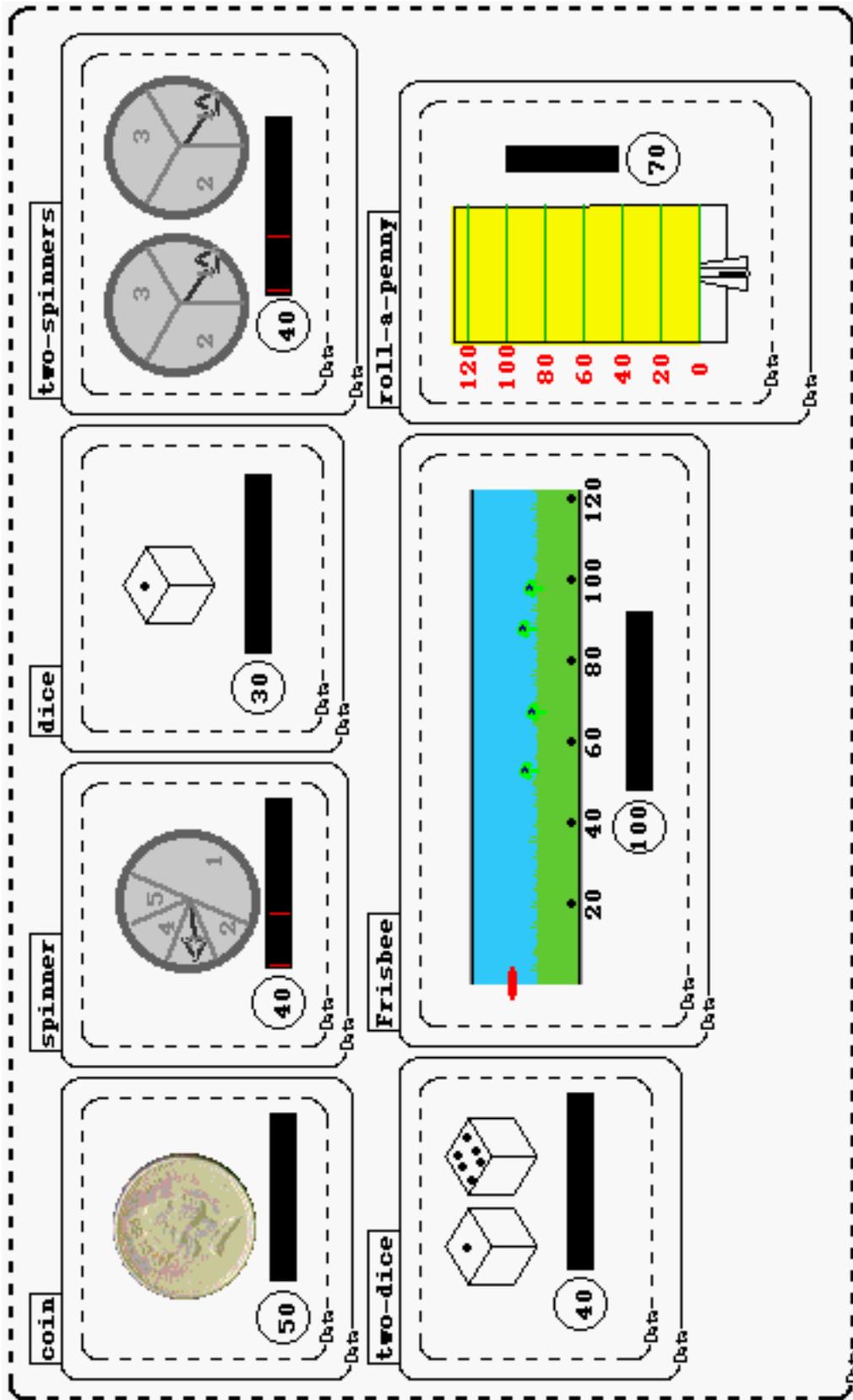


Figure 1. Chance-Maker's gadgets

Any gadget can be activated using the strength bar. When activated the gadget spins and turns much like its everyday equivalent. For example, Figure 2 depicts the strength bar for the TWO-DICE gadget as a solid black bar with a circular switch at one end. Imagine the child controlling the strength by allowing a tube (the black bar) to fill with a red fluid until the switch is clicked. The strength of the throw, 40% in this case, is represented by the amount of red fluid. Clicking directly on one of the dice causes the two dice to be thrown with the same strength as last time. When the dice are clicked, they roll ‘dice fashion’, indicating the resulting combination on their top surfaces.

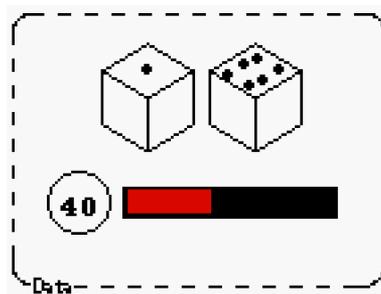


Figure 2. The Two-Dice gadget

Nevertheless, the microworld would be pointless if all it did was to simulate what can be achieved in the everyday. The gadgets can in fact be opened up to reveal further tools that are not accessible in conventional environments. Figure 3 illustrates the tools available when the TWO-SPINNERS gadget is opened up.

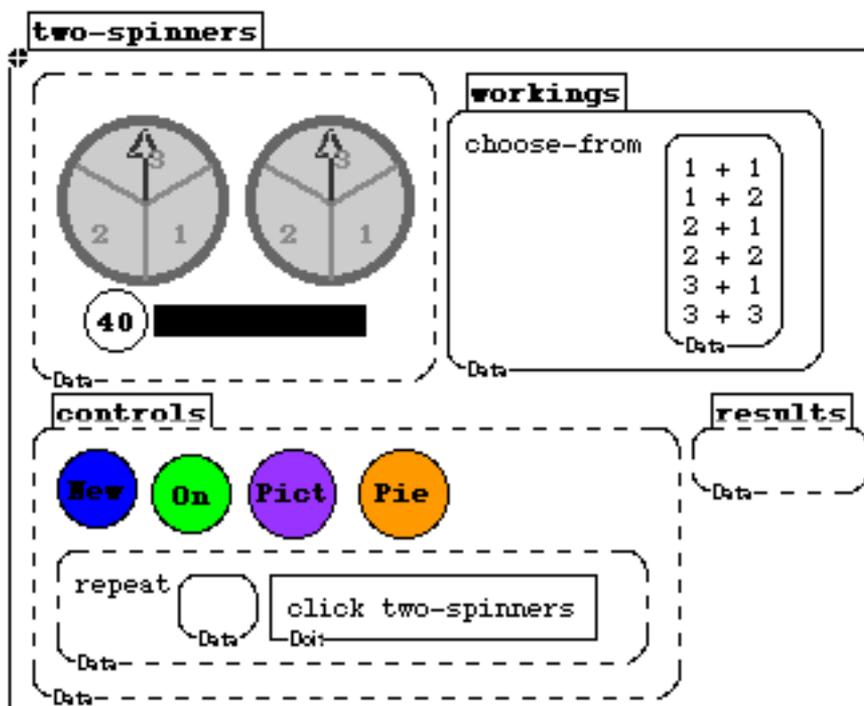


Figure 3. The tools inside the TWO-SPINNERS gadget

The outcome resulting from activating the TWO-SPINNERS gadget is controlled by its workings box, which captures the mathematical core of how the gadget works. The workings box can be edited by the child. In the default form as shown, the gadget contains only 6 of the 9 possibilities. The format of the workings box is intended to focus attention on the various combinations that might make up any particular total. One total, five, is not represented at all in the default version.

Figure 3 also shows the repeat tool (Logo-style), into which a child can enter the number of trials to be carried out. The ON/OFF button allows the graphics to be toggled -- typically the children switch the graphics off to save time when repeating many trials. The NEW button allows children to begin a new experiment with no previous results stored; the results box simply lists all results since the NEW button was last clicked. By clicking on one corner of the results box, a chart of the results is displayed instead of the list. Two types of chart are possible (see, for example, Figure 4, boxes B and E), generated by clicking on the PIE or PIC buttons. The pie chart shows the proportions of the results for each outcome. The pictogram is not automatically scaled -- differences between the number of results for each outcome are often highlighted -- though there is a feature which the child can use to scale the pictogram. By increasing the scale, differences between rows in the pictogram will be reduced proportionately. The TWO-SPINNERS gadget was introduced in later iterations as a less complex version of the TWO-DICE gadget, which contains similar tools; the workings box of the TWO-DICE gadget contains by default six of the 36 possible outcomes.

The children were challenged to identify which gadgets they thought were working properly, and to use the tools provided to mend those which they thought were not working properly. The design of the task was itself an important structuring resource since it provided a sense of purpose for activity with the gadgets and provoked interaction with the workings boxes. In this final iteration, the children worked on the gadgets in the following order: COIN, SPINNER, DICE, TWO-SPINNERS, TWO-DICE though the data presented here focuses on the latter two.

#### Local Resources for the Total of Two Dice

I have discussed elsewhere (Pratt, 1998; Pratt & Noss, 1998) the evidence that the children, in interacting with the COIN, DICE and SPINNER gadgets, articulated four local resources for randomness. Three of these, unpredictability, unsteerability and irregularity, have already been mentioned. The fourth was fairness.

According to this local internal resource, a phenomenon would be seen as random when the appearance looked symmetrical. In that study, the local resources were found to be used interchangeably, moment to moment, as the children sought explanations for the behaviour of the gadgets. Sometimes apparently contradictory resources appeared to be held simultaneously. For example, the children would see the COIN as random because of its unpredictability but would regard a non-uniform spinner as not random, basing this on the spinner's unfairness and ignoring its unpredictability. These local resources seemed to have been abstracted directly from everyday experience.

In this study, I propose to focus on the work of two children, Anne and Rebecca, each ten years of age. In this way, it will be possible to study in detail the emergence of new resources for making sense of the total of two spinners and two dice. Later I will discuss some of the main variations apparent in the clinical interviews with other children in the final iteration.

In the pre-interviews, both Rebecca and Anne declared that no total for two dice was harder or easier to obtain than any other. The root of this direct affirmation of Lecoutre's equiprobability bias can be identified when we look more closely at their reasoning. Anne intuited that the totals must be equally likely because the dice were (individually) fair and so the combination of them must be fair.

1. Anne: Because you can't estimate what number you'll get because they're all fair, both the numbers are fair.

Anne's articulation of the equiprobability bias (line 1) in relation to the total of two dice stems from the fairness local resource. Fairness was in general a strong discriminator of whether a gadget was working properly. Fairness appeared in the context of the appearance of the gadget, of the pie chart or pictogram of results, and (later) of the workings box. Rebecca also saw the total of two dice as equiprobable but, in her case, this resource had a different source.

2. Rebecca: Cos it's random, you can't control which number it lands on.

Rebecca applied the unsteerability resource (line 2) to the total of two dice. These resources for two dice were also apparent in their early interactions with the TWO-SPINNERS gadget (lines 3-7).

3. Researcher: If these were real spinners, and we can get any total between 2 and 6, do you think there is any total which is harder to get than the others; any total that's easier to get than any others?

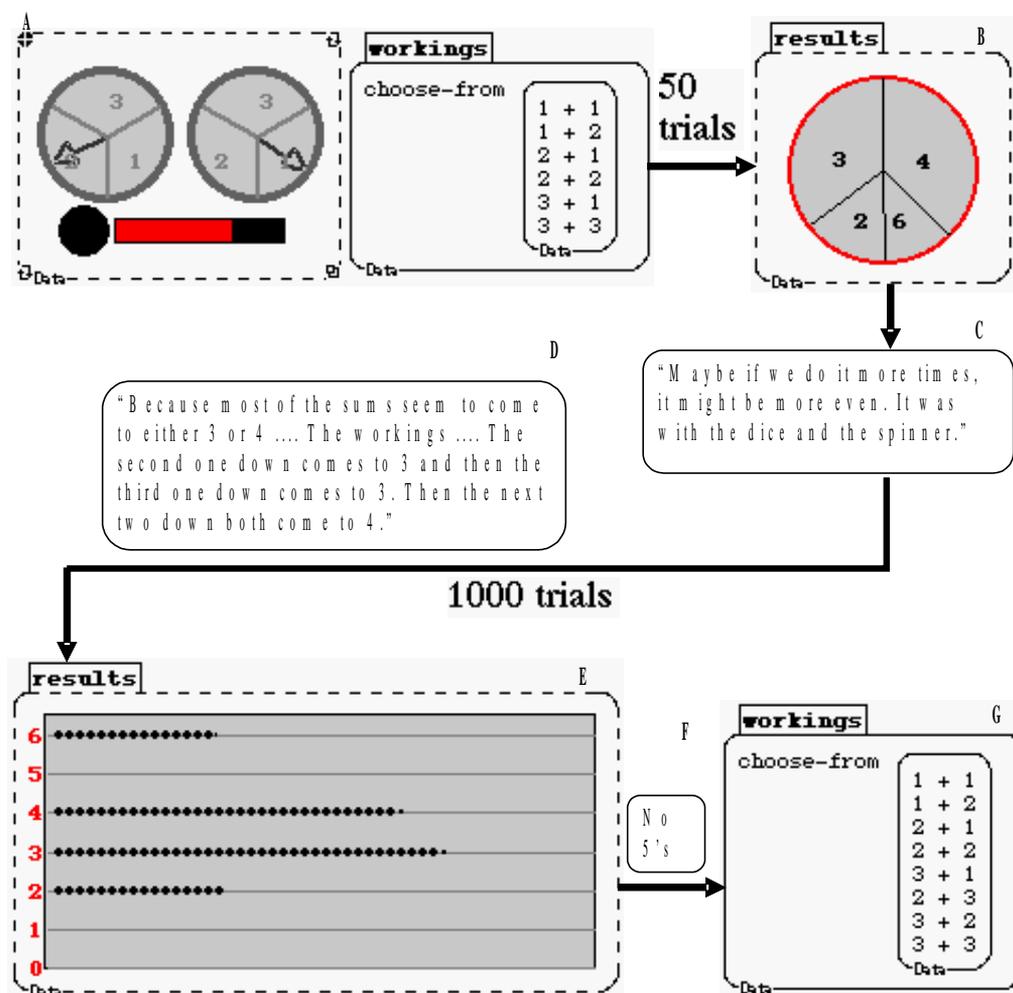
4. Anne: What it was in real life? (I confirm.) No.
5. Rebecca: There's a 50 / 50 chance of getting any total.
6. Researcher: So you think all the totals are equally easy or hard to get.
7. Anne and Rebecca: Yes.

Leading up to that point, the children had already encountered the COIN, SPINNER and DICE gadgets through which they had constructed new situated abstractions (reported in detail in Pratt, 1998). I will point out these previously constructed global resources as they occur.

#### Global Resources for the Total of Two Spinners and Two Dice

##### Using the TWO-SPINNERS Gadget

In Figure 4, I set out a schematic representation of Anne and Rebecca's early interactions with the TWO-SPINNERS gadget, based on the trace in Table 1 and their case analysis.



*Figure 4. Anne and Rebecca's early interactions with the TWO-SPINNERS gadget*

In the top-left-hand corner of Figure 4 (box A), we see how Anne and Rebecca began with 50 trials of the default version of the TWO-SPINNER gadget. They generated the pie chart (box B) and felt that there were too many 3's and 4's. It was clear at this stage that Anne and Rebecca expected the pie chart to look even, an interpretation consistent with their equiprobability bias.

In attempting to correct the overabundance of 3's and 4's, Rebecca applied a global resource abstracted from her prior work. I have described elsewhere (Pratt, 1998; Pratt & Noss, 1998) the evidence for these new resources, referred to as situated abstractions. One such situated abstraction is schematised as 'the number of trials controls the appearance of the pie chart'. In their interactions with the COIN gadget, the children had constructed new resources in which the appearance of the pie chart was a direct consequence of the number of trials used. This situated abstraction typically took the form of 'the higher the number of trials, the more even is the pie chart'. Rebecca (box C) recalled this situated abstraction and conjectured that it might explain the

behaviour of the TWO-SPINNERS gadget.

They carried out 1000 trials of the gadget. Whilst they waited for the pictogram to appear, it became clear that both girls believed that the 3's and 4's would in fact appear more often. This was not now based merely on the results of their early trials but on some analysis of the workings box (box D). In their prior work with the SPINNER gadget, Anne and Rebecca had already constructed the situated abstraction that 'the workings box controls the evenness of the pie chart' -- the expectation was that, by making the workings box fair, the pie chart would appear to be fair. It was not yet clear whether Anne and Rebecca's observation of 'unfairness' in the workings box would cue this prior situated abstraction.

The pictogram in fact showed that the total of 5 had not appeared (box E). Anne suggested that there were no numbers which made 5 (box F), but Rebecca pointed out that you could have a 2 and a 3 but this was missing from the workings. They edited the workings to include both 2+3 and 3+2 though they failed to consider 1+3. Rebecca explained:

8. Rebecca: because the first number is representing the first spinner, so you have to have it both ways.

In Figure 5, I schematise the subsequent interactions with the TWO-SPINNERS gadget.

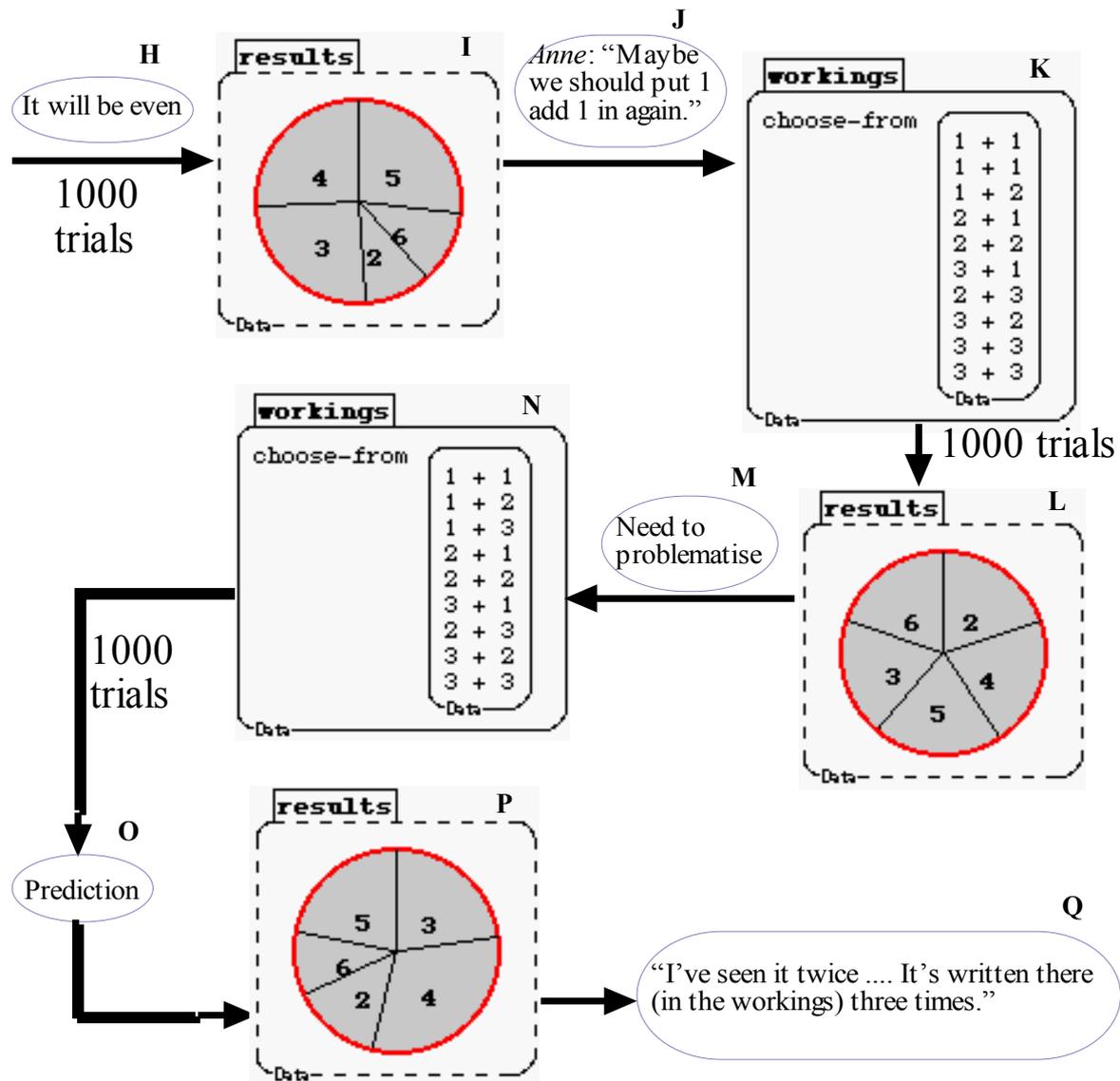


Figure 5. Anne and Rebecca's further interactions with the TWO-SPINNERS gadget

The story continues from box H in the top-left-hand corner of Figure 5. Anne and Rebecca repeated 1000 new trials with these new workings. Anne predicted that the pie chart would be even (box H), (presumably on the basis that the spinners were fair). The pie chart showed unequal sectors with least 2's (box I).

The next interchange revealed their current thinking. Anne wanted to add another 1+1 in order to make the pie chart fair (box J), which prompted the following exchange:

9. Rebecca: No, because it's already been done so it would just be the same.

10. Anne: Yes, because some of them have got the same again.

11. Rebecca: Yes, maybe we should actually put some of them in again, because then there's more chance

of them coming out more even.

Anne saw the insertion of another 1+1 as a way of equalising the sectors of the pie chart. Rebecca was torn between her resource that the different totals should be equally likely (because they could not be controlled) and her analysis that the doubles, unlike the non-doubles, did not need to be repeated in the workings. It was now clear that both girls believed strongly that the different totals should be equally likely, confirming previous evidence; this belief was so strong that they sought to fix the workings to make the pie chart come out even.

I decided to intervene to offer support to Rebecca's view. I asked Anne what she meant when she said that some of them were the same.

12. Anne: Well, 1 and 2 and 2 and 1 are the same ... they come to the same number.

13. Researcher: They come to the same total, but are they the same as far as the spinners are concerned?

14. Rebecca: No they are not. Because, the second one down, that number (pointing to the 1 of 1+2 in the workings box) refers to that spinner (pointing to the first spinner), and that number (pointing to the 2 of 1+2) refers to that spinner (pointing to the second spinner). So, say, if that one (the first spinner) lands on 1 and that one (the second spinner) lands on 2, it would be three. And if that one (the second spinner) lands on 1 and that one (the first spinner) lands on 2, it would be three as well.

15. Anne: Exactly ... I think we should add that one (pointing to the 1+1) and that one (pointing to the 2+2) again because then we get more of a chance of getting them.

16. Researcher: But then you would be putting in 1 plus 1 twice.

17. Anne: Yes, because 2 doesn't come up as much, does it?

18. Rebecca: So maybe if we do that.

My intervention was rejected by Anne, further emphasising the strength of her conviction that the pie chart should be even. They began to edit the workings. At one point Rebecca was about to add another 2+2 as well as a 1+1, but Anne pointed out that they already had lots of 4's. They finished up with the workings in box K.

The inconsistency of adding an extra 1+1 and not an extra 2+2 to the workings was not a concern for Anne since her aim was simply to equalise the likelihoods of the different sectors, and hence make the gadget work properly in her view. After 1000 trials, the pie chart was fairly even (box L). I needed to re-problematise the

situation so that Anne and Rebecca might question their solution (box M). The following interaction was critical.

19. Researcher: What we don't know for sure is whether that is how real spinners would behave. I think what you need to try and do is justify why you should have 1 plus 1 in there twice over.

20. Rebecca: Because everything else has two ways of coming except maybe 2 plus 2.

21. Researcher: But in reality, does 1 plus 1 have two different ways of coming?

22. Rebecca: I think it is more fair because the pie chart looks roughly even and before there were barely any 2's and barely any 4's ...

23. Researcher: I think you have certainly made it more fair. What I am not convinced about is that you have made it more like real spinners would be ... maybe with real spinners that would not be the case.

24. Anne: Oh, yes, mmm.

25. Researcher: You see, I am not sure you are being fair by putting 1+1 in twice.

26. Anne: We don't want it to be even. We want it to work like a real spinner.

My intervention aimed to suggest an alternative way of thinking about fairness by suggesting that it might be regarded as unfair to represent the same outcome twice over, especially as we wanted to make the two-spinners gadget behave like real spinners. I hoped that my intervention might raise the possibility that real spinners might not be fair in the sense that all totals should be equally frequent. It also raised the idea that fairness could instead be construed in terms of all the possible combinations being represented equally often. In Anne's last comment (line 26), it is important to know whether 'working like a real spinner' for her now means reviewing the combinations and making them fair or whether she is still wedded to the idea that the totals should be represented equally.

Anne and Rebecca immediately and spontaneously removed the 1+1, 2+2 and 3+3 from the workings, suggesting that they had recognised that the unfairness of including 1+1 twice extended to the other doubles. They did not immediately pick up the need to include 1+3, which only occurred after my intervention 'what about 1+3?' (box N). This may have been an accidental omission or it may indicate that their thinking about the fairness issue had been limited to doubles. Unfortunately, in hindsight, my intervention was too clumsy to extricate these two possibilities.

The girls generated 1000 trials and I asked them to predict what the pie chart would look like (box O).

27. Anne: A bit uneven ... because 1 and 1 has only got once, because that is what a real spinner would be like. And the rest has got like double number and it can make different numbers.

28. Rebecca: I think maybe ... 2 won't come up as much and 6.

The pie chart showed most 4's and least 2's and 6's, with slightly less 6's than 2's (box P). For the first time, Anne and Rebecca suggested that the two-spinner gadget should not generate an even-looking pie chart. Though Anne refers here only to the 1+1 case (line 27), the easiness with which they had previously amended 2+2 and 3+3 in the workings box suggests that Anne was in fact using 1+1 in a generic way to refer to all doubles. It follows from this interpretation that when she refers in the same line to 'the rest' she means the non-double combinations, which do indeed have two differently ordered representations. The interpretation is confused by Anne's reference to 'double numbers' which I am convinced does not refer to doubles but to non-doubles which are represented twice.

I looked for an explanation from them for the size of the 4 sector in this pie chart. Rebecca referred directly to the frequency of the total 4 in the workings box (box Q).

29 Anne: I've seen it twice ... It's written there three times ... Because it's got more of the numbers. It's got like three different numbers so it's coming up much more.

30 Researcher: Now, how do you think this compares to doing it with two real spinners?

31 Anne: Probably it would be about the same because we are trying to work it as a real spinner, and we've got the same sort of numbers.

32 Rebecca: I agree with Anne.

33 Researcher: So do you think the different totals, 2, 3, 4, 5, 6 are all just as easy to get or all just as hard to get. Or is there one that's easy to get?

34 Anne & Rebecca: (overlapping) 4 is easier to get.

35 Researcher: Is there one that is hard?.

36 Anne & Rebecca: 6.

37 Researcher: Just the 6?

38 Anne & Rebecca: 2

I take this as evidence that Anne and Rebecca were articulating a new situated abstraction, which could be schematised as: The more often a total is represented in the workings box, the larger will be its sector in the pie chart. But the discussion continued:

39 Rebecca : But 6 seems harder because it is smaller ....

40 Anne: Yes, much smaller.

41 Rebecca : Less of 6.

Although, Anne and Rebecca have constructed a situated abstraction based on the frequency of totals in the workings box, they nevertheless differentiate between 6 and 2 on the basis of the appearance of the pie chart. The new situated abstraction based on frequency of totals in the workings box has not replaced the resource based on the appearance of the pie chart. Both can be seen as resources available for sense-making and it is not clear which has higher priority.

I suggested that they repeat the experiment to see what happens. As we awaited the new pie chart for 1000 trials, the following discussion began.

42 Rebecca: There seems to be less ways of getting 6 and 2 than there are of 4, 5.

(The pie chart appears with a larger sector for 6's than 2's.)

43 Rebecca: The 2 is smaller this time than last time. Because last time 6 was smaller.

44 Researcher: So do you think 2 is easier to get, just the same, or harder to get than the 6?

45 Rebecca: Just the same, I think, because last time 6 was slightly bigger than 2 and this time... the other way round, 2 was bigger than 6 and this time 6 is bigger than 2.

46 Anne: Yes, I agree.

47 Researcher: What do the workings tell you about 2 and 6?

48 Rebecca: There's only one way of getting them, and there's two for 3 and 5, and three ways of getting 4.

49 Researcher: So, if we were going to play a game, in which we had two spinners like this, and you are going to bet a pound and I am going to bet a pound. And you're going to bet that a total of 4 comes up, and

I'm going to bet that a total of 6 comes up. Would you take that bet?

50 Anne: No ... well, sort of ... I wouldn't bet as much as a pound.

51: Researcher: How much would you bet?

52 Anne: 20p.

53 Rebecca: 2p.

54 Researcher: If we were to bet 2p the other way round that I'm betting on a 4 and you're betting on a 6, would you take that bet?

55 Rebecca: No.

56 Anne: Definitely not.

The discussion suggests that the girls were now placing quite a high priority on the situated abstraction based on the frequency of the totals in the workings box rather than on the appearance of the pie chart about the likelihood of the various totals (lines 42 and 48) and that the appearance of the pie chart is seen as less convincing (lines 43, 45, 46). There appeared to be some commitment to the situated abstraction (lines 49 to 56).

#### Using the TWO-DICE Gadget

At this point, Anne and Rebecca turned their attention to the TWO-DICE gadget. Figure 6 traces their interactions.

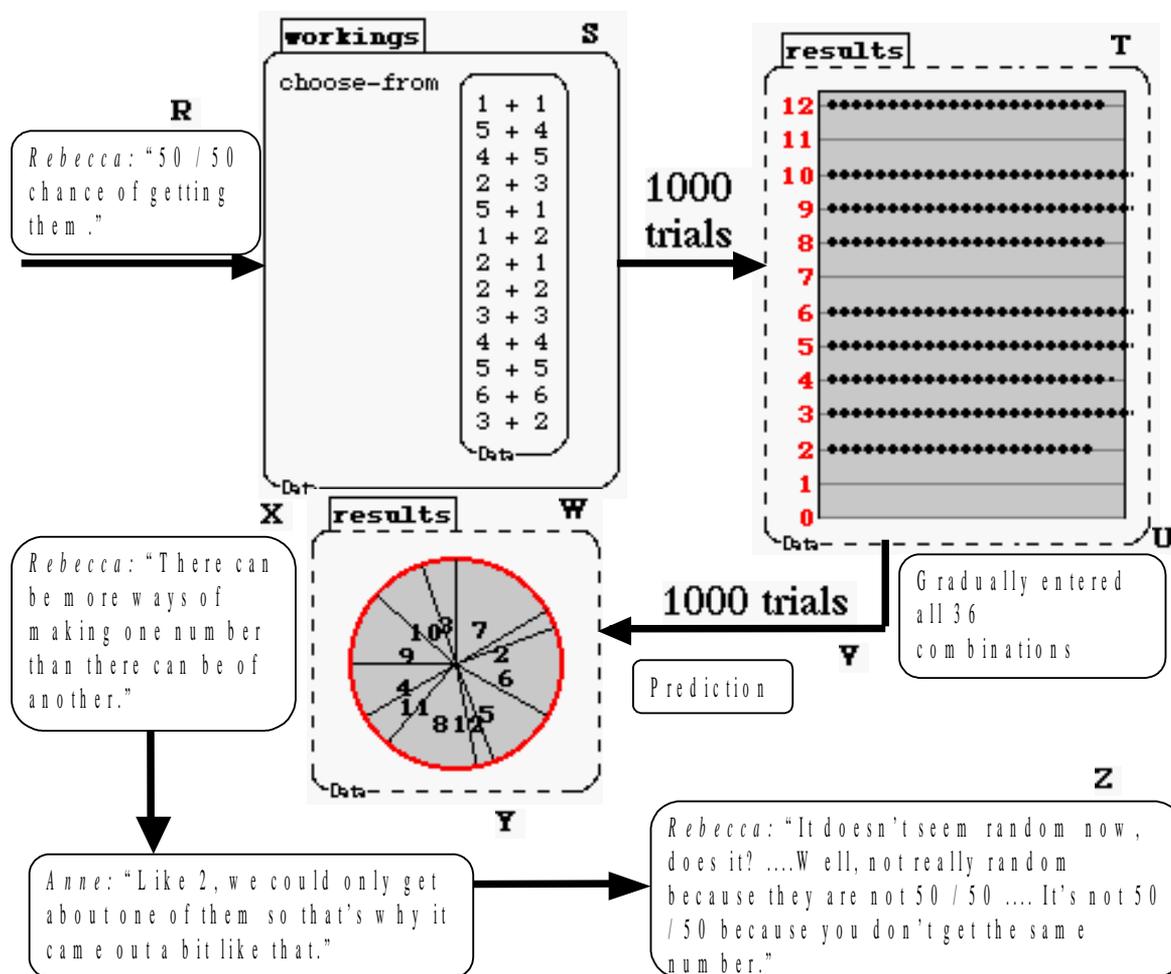


Figure 6. Anne and Rebecca's interactions with the TWO-DICE gadget

It was clear from their early discussions about the TWO-DICE gadget that they believed that there were missing data in the workings.

57. Rebecca: There's lots of different ways ... there's probably more ways of making 6 ....

58. Anne: Ah, 5 add 1 is missing to make 6.

This discussion resulted in some modification of the workings box with, for example, 5+1 being added. At this point I wished to find out whether their thinking about the totals of two dice had changed in the light of their situated abstractions from the two-spinners gadget.

59. Researcher: If we were shaking two real dice, do you think all the totals you could get are just as easy, just as hard, or do you think some totals are easier than others, harder than others?

60. Rebecca: 50 / 50 chance of getting them. (Anne agreed.)

61. Researcher: So you think they are all about the same chance?

62. Both: Yes.

Anne and Rebecca clearly articulate the equiprobability bias with respect to the total of two dice. This is surprising given the level of analysis that they had been providing for the total of two spinners. The global resources constructed from the TWO-SPINNERS work were not cued by this new situation. Instead, local resources articulated in the pre-interviews were cued.

Anne and Rebecca then noticed other outcomes missing and amended the workings box accordingly (box S). After 1000 trials, they generated a pictogram showing no 7's nor 11's (box T). With some support from me, aimed at helping them to be systematic, Anne and Rebecca incorporated all 36 combinations into the workings box (box U) and decided to try out the new situation with 1000 trials. Anne made a prediction about the pie chart:

63. Anne: Fairly even? ....Some of the numbers might not be because there's not as much as the other number.

64. Rebecca: Maybe roughly even because now that we have got all the sums. I'm not too sure at the moment.

65. Anne: I think some will be a bit less because they haven't got as much as the others ... because some of the numbers will not be the same, will be less, because we didn't find enough sums for them ... like 1 add 1.

66. Researcher: Can you give me an example of one that had a lot of different ways of getting it.

67. Anne: Seven.

Through their interactions with the workings box, Anne recognised that some totals were represented more often than others, although Rebecca still thought the pie chart might turn out to be even. Though their expectation had been that, by inserting the extra data into the workings box, the pie chart would appear more even, the editing process had alerted Anne to the unequal representation of different totals. Their previous experience with the TWO-SPINNERS gadget offered the possibility that the pie chart might be uneven as a result. There were then two internal resources available to Anne and Rebecca:

- the situated abstraction that the sectors in the pie chart were determined by the frequencies of the corresponding totals in the workings of the two-spinners gadget, or

- the local resource that each individual dice was unsteerable and fair and so the various totals were equally likely.

The pie chart showed most 7's and least 2's and 12's (box W). Rebecca's first reaction was that there might still be some missing outcomes in the workings box but then observed (box X):

68. Rebecca: Ah, I bet there are various ways of making a number. There can be more ways of making one number than there can be of another.

By inspecting the workings box, the girls identified that the 12's, 2's and 3's did not have many ways whereas the 7 had the most. I asked why then the pie chart had that appearance. Anne replied (box Y):

69. Anne: Because some of the sums we put as more. Like 2, we could only get about one of them so that's why it came out a bit like that.

Rebecca's final comment (box Z) showed the extent to which she was constructing new resources for the behaviour of the TWO-DICE gadget.

### Variations

Before summarising and discussing the significance of these findings, I will first briefly discuss how the episode with Anne and Rebecca compared to the work of other children in the final iteration. The description of Anne and Rebecca's work is evocative of the work with the compound gadgets for seven out of the eight pairs of children in the final iteration. These seven pairs all forged new but similar situated abstraction for the behaviour of the compound gadgets.

The one exceptional pair was Donna and Rose, who did not construct an alternative resource for fairness. For them, the visual image of the symmetry of the spinners was so powerful that phenomena were interpreted in terms of that resource. If the data did not conform to that resource then they had to find an explanation for why the phenomenon was misbehaving. In contrast, the other pairs were able to construct a new resource for fairness, defined in terms of the representation of all the possible simple events equally in the workings box. This was the meaning that eluded Donna and Rose. For them, the workings contained unfair representation of the total 3 since it appeared twice in the workings (1+2 and 2+1) whereas other totals, like 2, were only represented once. For some reason, they never noticed that the total 4 was also represented twice. Donna and Rose did not recognise 1+2 and 2+1 as distinct. They never generated an even-looking pie chart, and so they

searched for other explanations in phenomena such as the strength of throw.

Several pairs constructed similar situated abstractions to those of Anne and Rebecca for the behaviour of the compound gadgets but by a different route. These children tended to favour the use of both the pictogram and the pie chart tool. In fact, they increasingly found the pictogram more supportive in the case of the TWO-SPINNERS and TWO-DICE gadgets. For example, the following interaction took place between Neil and Gurdev just after generating a pictogram from 1000 trials of the TWO-DICE gadget.

70. Neil: 7 ... (tracing the workings with the mouse), 1 plus 6, ....ah, I know.

71. Gurdev: There's loads of 7's.

72. Neil: 3 plus 4, 1 plus 6, 3 plus 4, 4 plus 3, 5 plus 2, 6 plus 1.

(After some further discussion about whether 7 should be bigger or not ....)

73. Neil: But I think that ... wouldn't they all be equal? We've put all the way up 6. I thought we would get all equal amounts.

74. Researcher: What would be all equal amounts?

75. Neil: All of the numbers.

76. Researcher: The same number of 2's as 3's as 4's as 5's as 6's, and so on?

77. Neil: Actually no, because 7 is a higher number than 6 and on there you can add it up more. Like if it was 4, you can't add 6 to something because it is too high.

A few minutes later Neil and Gurdev repeated another 1000 trials. The pictogram almost showed the classic triangle (Figure 7).

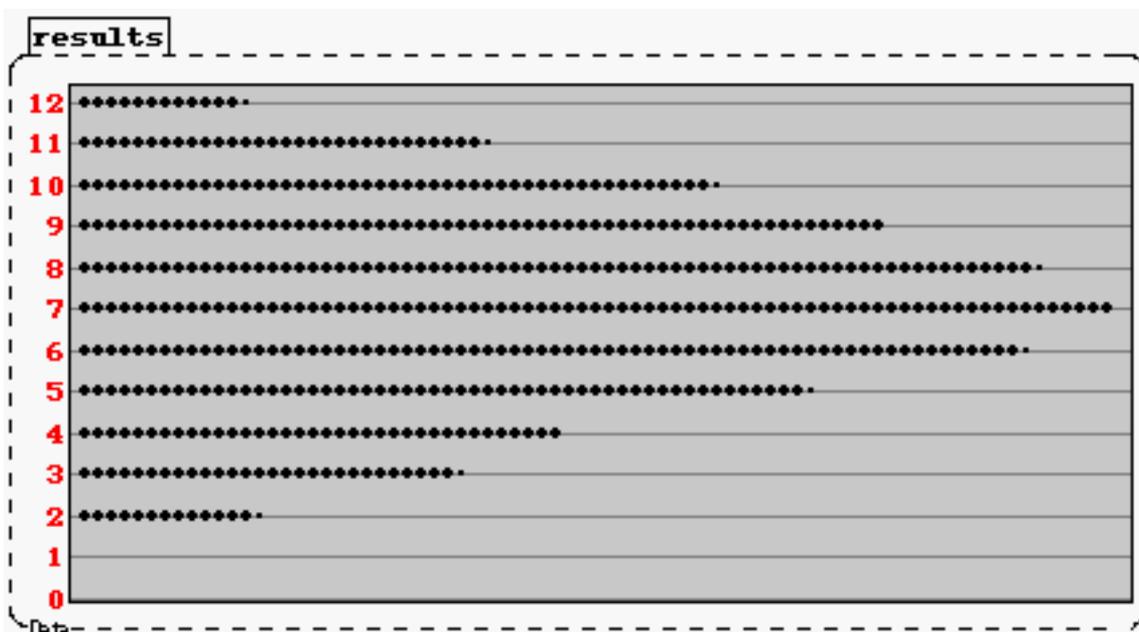


Figure 7. "The pictogram almost showed the classic triangle"

78. Neil: I thought so. The 8 is bigger now. Look it goes like that, kind of. They all go up in 1 except, if that (the 7 perhaps) was one more, it goes like that, doesn't it (showing a triangle pattern with his hands)? ... It's something to do with the workings ... Is it because the numbers in the workings are always one step ahead.

(I asked about the total of 11)

79. Neil: 6 plus 5 and 5 plus 6. Two ways.

(I asked how that compares to 3's.)

80. Neil: We've got 1 plus 2, 2 plus 1. You can only do it twice. There's no other way.

81. Gurdev: Oh, they keep on going up about ... oh, I get it now ... Just start from the 2's and the 12's. They start off small. Then it goes to 3's and 11's, they get a bit bigger. Then a bit bigger, a bit bigger, ....

82. Neil: Yes, that's what I was trying to explain.

83. Researcher: When you say a bit bigger, a bit bigger, you are talking about the number of combinations, or calculations as you put it, that can do it, not the actual results but what's in the workings.

84. Both: Yes.

Neil and Gurdev had constructed a situated abstraction that the workings are stepped like the triangular pattern in the pictogram.

## Discussion

I wish now to discuss in turn the original three research questions for this study before considering how this evidence illuminates the notion of webbing and diSessa's model of conceptual change.

### What are the internal resources which children use to make sense of the total of two dice?

The children came to this activity with a variety of local resources, based on symmetry and experience of short-term behaviour, that the totals should be uniformly distributed. Though there was little evidence in this study of biases like availability, Lecoutre's equiprobability bias was very strongly articulated, and in the case of one pair, Donna and Rose, remained the dominant way of making sense of the behaviour of the compound gadgets. The equiprobability bias was associated with local resources such as unsteerability and fairness (lines 1, 2 & 5; 45 to 48).

In addition, the children made explicit use at various times of global resources constructed from earlier work with the COIN, SPINNER and DICE gadgets, namely 'the number of trials determines the fair appearance of the pie chart' (box C) and 'the workings box determines the fair appearance of the pie chart' (box J and lines 9 to 11). The children's own investigations with the TWO-SPINNERS gadget offered evidence which neither their local resources nor their recently acquired global resources could render consistently meaningful. Nevertheless, these helped to shape the emergence of new resources for the behaviour of the compound gadgets.

### In making sense of the total of two dice, what situated abstractions are forged through the webbing of these internal resources and the external resources embedded in the setting?

The children constructed a new situated abstraction, a refinement of the 'workings box controls the pie chart' situated abstraction. This situated abstraction took the form that the frequency of representations of a total in the workings box of the TWO-SPINNERS gadget controls the size of its sector in the pie chart (box Q and lines 27 to 28). They also constructed the new situated abstraction that the frequency of representations of a total in the workings box of the TWO-DICE gadget controls the size of its sector in the pie chart (boxes X, Y and Z and lines 40 to 41). Some of the children constructed the new situated abstraction that the pictogram is stepped because the workings box is stepped (lines 50 to 56).

### What are the features of the webbing process which determine the extent to which these situated abstractions become tools for the forging of new connections in related activity?

I have reported elsewhere that newly acquired situated abstractions do not get automatically cued in different circumstances (Pratt, 1998). The situated abstraction, ‘the higher the number of trials, the more even is the pie chart’, which had previously been constructed during activity with the COIN and SPINNER gadget, used spontaneously by Anne and Rebecca soon after they began working with the TWO-SPINNERS gadget (box C), providing evidence that by this stage this situated abstraction had taken on greater priority. On this occasion of course, Anne and Rebecca had over-generalised its usage. Learning to discriminate the limitations of intuitive knowledge is part of the tuning towards expertise. In contrast, when the two children began working with the TWO-DICE gadget, their predictions were initially articulated in terms of the equiprobability bias, and not in terms of the newly acquired situated abstraction, ‘the frequency of representations of a total in the workings box controls the size of its sector in the pie chart’. Either this is because the claim that the situated abstraction had been constructed is false or its level of priority was insufficiently low for it to be cued in new circumstances to which it might not apply. The level of commitment at the end of the TWO-SPINNERS work to this situated abstraction suggests to me that the latter explanation is more plausible.

The tools and structures within the setting played a critical role in enabling them to construct new ways of making sense of the compound gadgets. We saw in Anne and Rebecca’s work the construction of new situated abstractions for the behaviour of the compound gadgets. These new resources emerged through the co-ordination of local resources and prior global resources. Why did this co-ordination take place? The mending activity shifted attention from the behaviour at a surface level of the gadget to the behaviour as influenced by the workings box. At this point, the children’s attention was more easily pointed towards the various combinations of possible outcomes. The role of the researcher as participant observer was often critical at these moments. The opportunity to perturb children’s thinking, much as a teacher might do, was provided by the children’s need to engage with the workings box in order to make the gadgets behave like a real spinners or real dice. Attention on the combinations temporarily masked the randomness of the gadgets in a similar way to LeCoutre’s masking in her experiments on the equiprobability bias. During this period, fairness was reconstructed in terms of each combination, rather than each total, appearing once. Crucially, and in contrast to LeCoutre’s experiments, the design of this activity allowed the children to de-mask the experimental conditions at any time and under their control. The children were able to stress or ignore the combinations in the workings box. I believe that this element of control allowed the children to construct connections between the combinations in the workings box and the randomness of the gadget. The notion of fairness then was adapted to yield a situated abstraction in which each combination was represented once in the workings box; the totals

were then discerned by accumulation and counting within the workings box. This procedure forged a new connection between the workings box and the chart, often the pictogram, in which the accumulation of outcomes which represented a particular total controlled the appearance of the pictogram.

This case analysis offers an account of what Noss and Hoyles (1996) refer to as webbing between internal and external resources. In Figure 8, the webbing process, which enables this co-ordination is sketched out.

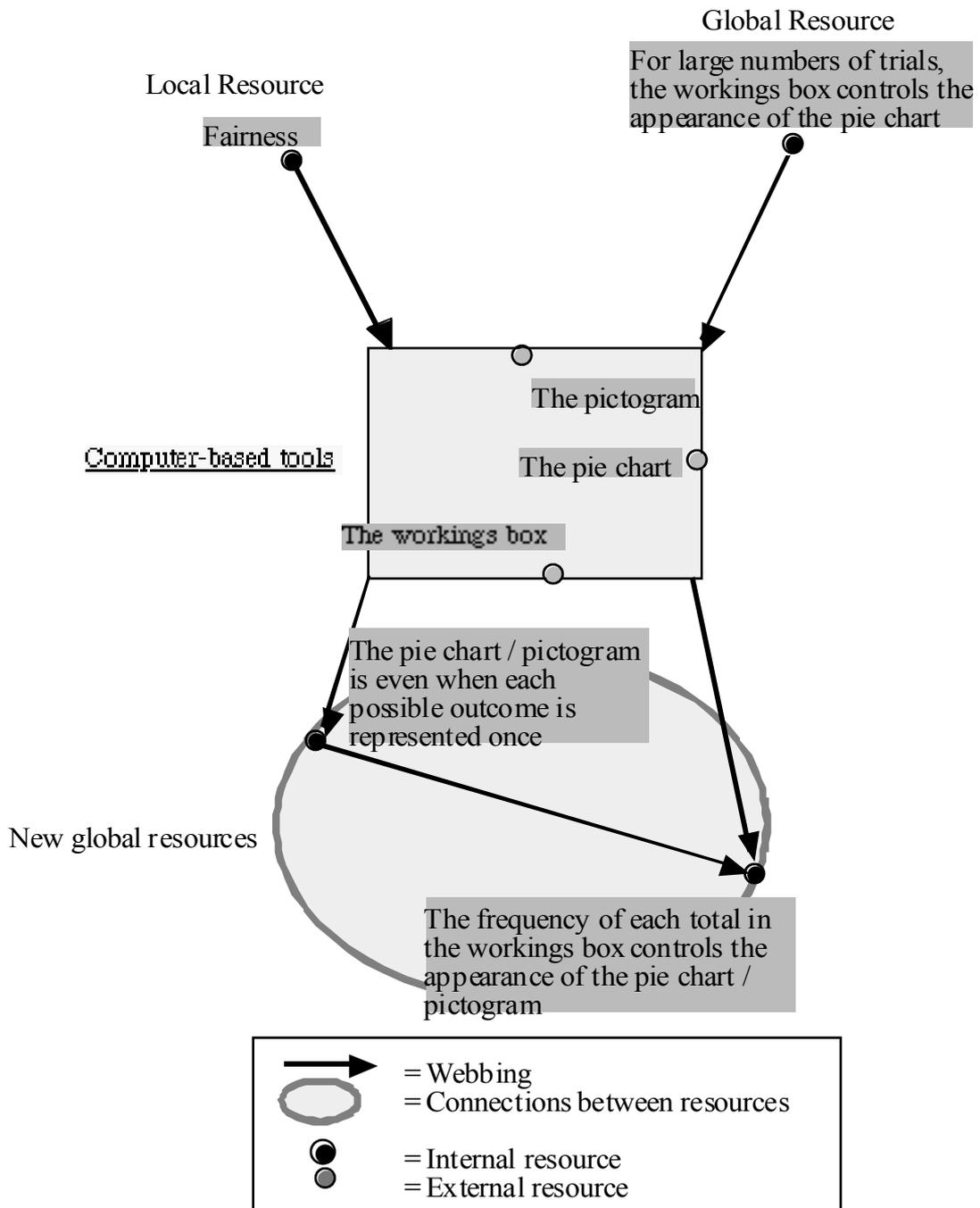


Figure 8. The construction of global resources for the behaviour of compound events

In Figure 8, activity with the TWO-SPINNERS or TWO-DICE gadgets cues the local resource of fairness and the situated abstraction constructed from previous activity for the predictability of the pie chart in the long term. These local and global resources are illustrated at the top of the diagram, and they are shown to be connected, because of the sense-making activity, with the computer-based tools. Out of this activity, new global resources for the behaviour of compound events emerge. First a situated abstraction, ‘the pie chart is even when each possible outcome is represented once’, is constructed. Further activity results in a new resource in which it is recognised that the frequency of the various totals in the workings box controls the appearance of the pie chart.

These new global resources do not replace prior resources, though they may gradually begin to dominate as their priority increases. The data demonstrates children are able to call upon local resources, prior global resources or recently constructed global resources as they attempt to make sense of novel situations. The Chance-Maker microworld provides a conjecturing environment in which children can experience the consequences of their beliefs. Within this sort of environment, it is possible to test out resources for making sense of the total of two spinners or two dice. Such explorations demand the use of whatever resources can be brought into play in order to explain behaviour. diSessa’s theory of conceptual change proposes that we hold concurrently multiple resources, in the form of p-prims. The evidence in this study points strongly towards the relevance of diSessa’s theoretical framework for the construction of resources for randomness in general and for the behaviour of the TWO-SPINNERS and TWO-DICE gadgets in particular. In a carefully designed environment, the choice of tools and resources can help to generate what diSessa calls ‘tuning towards expertise’, a process in which primitive resources, such as the local resources abstracted directly from experience, are re-structured and connected with new global resources.

Across activity with all the gadgets, I have observed young children placing increasing reliance on newly acquired, though situated, abstractions and discriminating the domain under which long-established local resources and recently constructed global resources are applicable. diSessa’s model attaches to p-prims two parameters; the cueing and reliability priorities determine the likelihood that any particular p-prim will be used under any specific circumstance. We can re-construct the episode with Anne and Rebecca as the bootstrapping of new p-prims for the behaviour of the TWO-SPINNER and TWO-DICE gadgets, followed by a period in which these new p-prims take on increased reliability as their consistency with the behaviour of the gadgets is maintained. New circumstances, such as when a new gadget is explored, or when two dice are used again in a board-game, may well cue the initial local resources, though we might predict that, as the reliability of the new

global resources is increased, so is the likelihood that these resources will be cued in novel, even everyday, situations. (I intend to elaborate in more detail in a future article the manner in which it is illuminating to apply diSessa's theory of conceptual change, based on the study of Physics students, to the emergence of resources for chance.)

This study provides evidence that our resources for making judgements of chance are dependent upon the tools and resources which shape their construction. Indeed, by providing new tools, it is possible to offer environments in which children can test out the reliability of long-established resources. I agree with Fischbein (1997):

In learning probability, students must create new intuitions. Instruction can lead students to actively experience the conflicts between their primary intuitive schemata and the particular types of reasoning specific to stochastic situations. If students can learn to analyse the causes of these conflicts and mistakes, they may be able to overcome them and attain a genuine probabilistic way of thinking. (p.104)

The Chance-Maker microworld offered the opportunity to recognise limitations in their local resources such as fairness, to reduce the level of their priority. By linking the frequency of totals in the workings box to the randomness of the gadget and the appearance of the charts, they began to place more reliance on the workings box as the arbiter of fairness. Environments like the Chance-Maker microworld can provide tools to enable the forging of new normative resources which might be applied in novel situations where a new gadget is being explored.

## References

- Biggs, J. B. & Collis, K. F. (1982). Evaluating the Quality of Learning: The SOLO Taxonomy. New York: Academic Press
- diSessa, A. A. (1983). Phenomenology and the Evolution of Intuition. In D. Genter & A. Stevens (Eds.), Mental Models. Hillsdale, New Jersey: Lawrence Erlbaum Associates.
- diSessa, A. A. (1993). Towards an Epistemology of Physics. Cognition and Instruction 10(2 & 3), 105-226.
- Fischbein, E. (1975). The Intuitive Sources of Probabilistic Thinking in Children. Dordrecht; London: Reidel
- Fischbein, E. (1982). Intuition and proof, For the Learning of Mathematics 3(2), 9-19.
- Fischbein, E & Schnarch, D. (1997). The Evolution With Age of Probabilistic, Intuitively Based Misconceptions, Journal for Research in Mathematics Education 28(1), 96-105.
- Hoyles, C., & Noss, R. (1993). Out of the Cul-De-Sac?, Proceedings of the Fifteenth Annual Conference of North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 1 (pp. 83-90).
- Kahneman, D., Slovic, P., & Tversky, A. (1982). Judgement Under Uncertainty : Heuristics and Biases. Cambridge: Cambridge University Press.
- Kahneman, D., & Tversky, A. (1973). On the Psychology of Prediction. Psychological Review 80(4), 237-251.
- Konold, C. (1989). Informal Conceptions of Probability. Cognition and Instruction 6(1), 59-98.
- Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993). Inconsistencies in Students' Reasoning about Probability. Journal for Research in Mathematics Education 24(5), 392-414.
- Lave, J. (1988). Cognition in Practice. Cambridge: Cambridge University Press.
- Lecoutre, M.P. (1992). Cognitive Models and Problem Spaces in 'Purely Random' Situations. Educational

Studies in Mathematics 23(6), 557-568.

Noss, R., & Hoyles, C. (1996). Windows on Mathematical Meanings: Learning Cultures and Computers. London: Kluwer Academic Publishers.

Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). Street Mathematics and School Mathematics. Cambridge: Cambridge University Press.

Piaget, J., & Inhelder, B. (1951). The Origin of the Idea of Chance in Children. New York: Norton.

Pratt, D. (1998). The Construction of Meanings In and For a Stochastic Domain of Abstraction. PhD Thesis, University of London

Pratt, D., & Noss, R. (1998). Expressions of Control in Stochastic Processes. L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee, & W. Wong (Eds.), Proceedings of the Fifth International Conference on Teaching of Statistics, Vol. 2 (pp. 1041-1047), Singapore.

Pratt, D. (1998). The Co-ordination of Meanings for Randomness. For the learning of mathematics 18(3), 2-11.

Speiser, R., & Walter, C. (1998). Two Dice, Two Sample Spaces. L. Peira-Mendoza, L. Seu Kea, T. Wee Kee, & W.-K. Wong (Eds.), Proceedings of the Fifth International Conference on Teaching of Statistics, Vol. 1 (pp. 61-66), Singapore: Voorburg: ISI Permanet Office.

Tarr, J. E. (1998). Middle school students' misuse of the phrase '50-50 chance' in probability instruction. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolbe, K. Norwood, & L. Stiff (Eds), Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. (pp. 401-406), ERIC Clearinghouse, Columbus, OH.

Tversky, A., & Kahneman, D. (1973). Availability: A Heuristic for Judging Frequency and Probability. Cognitive Psychology (5), 207-232.

Vidakovic, D. (1998). Children's Intuition of Probabilistic Concepts Emerging from Fair Play. L. Peira-Mendoza, L. Seu Kea, T. Wee Kee, & W.-K. Wong (Eds.), Proceedings of the Fifth International Conference on Teaching of Statistics, Vol. 1 (pp. 67-73), Singapore: Voorburg: ISI Permanent Office

Watson, J.M., Collis, K.F. & Moritz, J.B. (1997) The Development of Chance Measurement, Mathematics Education Research Journal 9(1), 60-82

## iFootnotes

Paradoxically though, as a mathematician, I act as if mathematical knowledge is decontextualised. The tension in this paradox underlies this study and will be the focus of a future article.

ii Professor Andy diSessa heads a team at the University of California, Berkeley, which is developing what is termed a computational medium, Boxer. This project is seen as extending the notion of literacy to a new domain, where users express themselves in various ways, including mathematically, in various modalities (graphic, literal, computational, ...) within the Boxer medium. Boxer is particularly well suited to iterative design because of the ease and flexibility with which one can reconstruct the interface.

iii This figure, and many of those later, will include the use of dashed and full lines. This should not be allowed to be a distraction to the reader. The use of different types of borders is a programming feature of Boxer related to the constraints on how data is passed from one box to others. In the context of this article, it is not important, and was completely ignored by the children in the study.